Topology and The Poincare Conjecture

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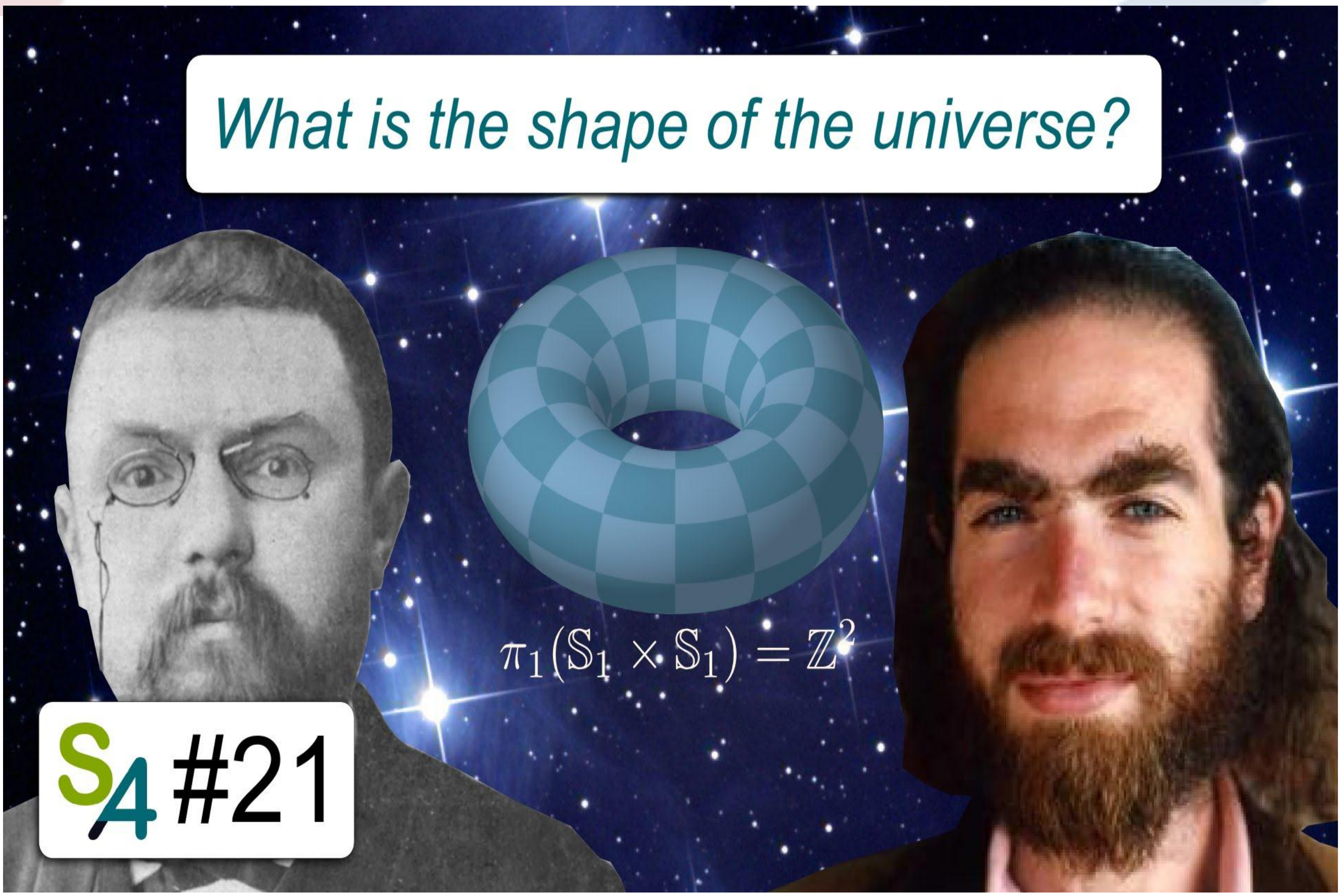


Introduction

A Russian mathematician Grigori Perelman, wore a rumpled dark suit and sneakers, and paced while he was introduced. Bearded and balding, with thick eyebrows and intense dark eyes, he tested the microphone and started hesitantly: "I'm not good at talking linearly, so I intend to sacrifice clarity for liveliness." Amusement rippled through the audience, and the lecture began. He picked up a huge piece of white chalk, and wrote out a short, twenty-year-old mathematical equation.

The equation, called the Ricci flow equation, treats the curvature of space as if it were an exotic type of heat, akin to molten lava, flowing from more highly curved regions and seeking to spread itself out over regions with lesser curvature. Perelman invited the audience to imagine our universe as an element in the gigantic abstract mathematical set of all possible universes. He reinterpreted the equation as describing these potential universes moving as if they were drops of water running down enormous hills within a giant landscape. As each element moves, the curvature varies within the universe it represents and it approaches fixed values in some regions. In most cases, the universes develop nice geometries, some the standard Euclidean geometry we studied in school, some very different. But certain tracks that lead downhill bring problems the elements moving along them develop mathematically malignant regions that pinch off, or worse. No matter, the speaker asserted, we can divert such tracks; and he sketched how.

The Poincare conjecture provides conceptual and mathematical tools to think about the possible shape of the universe. But let us start with the simpler question of the shape of our Earth. Any schoolchild will say that the Earth is round, shaped like a sphere. And, in these days of airplanes and orbiting spacecraft that can take pictures of our planet from on high, this seems utterly obvious. But, in times past, it was difficult to say with certainty what the shape of the world was.

Belief is one thing, but when did we really know, without any doubt, that the world was shaped like a sphere? We have seen that Columbus began to doubt the sphere theory, thinking of the Earth as pear shaped. And today we know that our planet is not a perfectly round sphere, but is flattened somewhat at the poles.

But, as we shall see presently, there are other more radical possibilities: the question of the Earth's shape is much more than a matter of bumpiness and flattened regions.

We have one final bit of housekeeping. We need to be precise about what it means to say that two manifolds are the same. As with everything else, this depends on one's point of view. Two objects can be same, or equivalent, in one sense, but different in another. In talking about shape, one is usually not concerned with features like size or distance that pertain to geometry, but about properties that are preserved under stretching and small deformations. Such properties belong to the domain of topology.

Let us say that two surfaces are the same topologically if the points of one can be put into one-to-one correspondence with the points of the other so that nearby points correspond to nearby points (such correspondences are called continuous). Two manifolds that are the same topologically are also said to be homeomorphic, and the one to one correspondence that establishes that they are the same is called a homeomorphism. Topology studies properties of surfaces (and other objects) that allow one to tell whether or not two surfaces (or other objects) are homeomorphic. Such properties are called topological properties.

Topological properties can be very different from geometric properties such as length and angle. Any two surfaces that can be deformed into one another by pulling and stretching (no tearing-because that can destroy continuity) are homeomorphic. Two spheres of different radii are homeomorphic. The surfaces in the figure below are all homeomorphic, and a topologist would view them all as spheres.

These are all homeomorphic to the two-dimensional sphere.

The problem of research:

- **If a compact 3D manifold M^3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3?**

1-Topology

Topology, sometimes referred to as "the mathematics of continuity", or "rubber sheet geometry" ,or "the theory of abstract topological spaces", is all of these, but, above all, it is a language, used by mathematicians in practically all branches of our science. In this chapter, we will learn the basic words and expressions of this language as well as its "grammar".

Definition (1;1):[16]

A topological space is a pair (X,T) where X is a set and T is a family of subsets of X (called the topology of X) whose elements are called open sets such that ;

- Ø,X belong to T, so they are open sets.
- $if\ \{O_i\}_{i=1}^{k} \subset T\ then\ \bigcap_{i=1}^{k} O_i \subset T.$
- if $\{O_a\}_{a\in A} \subset$ T then $\bigcup_{a\in A} O_a\ belongs\ to\ T$.

If x ∈ X, then an open set containing x is said to be an (open) neighborhood of x.

Example:

let X={a,b,c,d,e,f} andT={X,Ø,{a},{c,d},{a,c,d},{b,c,d,e,f}} then we notice that T is a topology on X because it satisfies the previous conditions.

Let X be any non-empty set and let T be the collection of all subsets of X, Then T is called the discrete topology on X, and (X,T) is called discrete space. Let U={X,Ø}, then U is called the indiscrete topology and (X,U) is said to be an indiscrete space.

lemma (1;1) : [16]

if X={a,b,c} and T is a topology on X with {a},{b},{c} belong to T, then T is the discrete topology.

- Proof:

  The set X has 3 elements and so it has 8 distinct subsets and they are {{a},{b},{c},{a,b},{b,c},{c,a},{a,b,c},Ø}, we are required to prove that each of these subsets is in T. We know that T is a topology on X so X,Ø are in T, and {a},{b},{c} are also in T. from the definition of the topology we know that the union of some sets in T belongs to T, so {a,b},{b,c},{c,a} belong to T, hence we are done.

Let us have these lemmas without proofs.

lemma (1;2): [16]

i. (1;2;1) Let X be an infinite set and T a topology on X. If every infinite subset is in T, then T is the discrete topology.

ii. Let T1 be a set and consist of N,Ø, and every set {1,2,3,…..,n} for n any positive integer then T1 is a topology. (PS: it is called the initial segment topology)
iii. Let T2 be a set and consist of N,Ø, and every set {n,n+1,n+2,……} for n any positive integer then T2 is a topology. (PS: it is called the final segment topology)
iv. Let T3 be a set and consist of R,Ø and every interval [-n,n] for n any positive integer, then T3 is a topology.
v. Let T4 be a set and consist of R,Ø and every interval (-n,n) for n any positive integer, then T4 is a topology.
vi. Let T5 be a set and consist of R,Ø and every interval [n,∞) for n any positive integer, then T5 is a topology.

(1;2) Open, Closed and Clopen Sets:[16]

Definition (1;2;1):

Let (X,T) be any topological space. Then the elements of T are said to be open sets.

And we call a subset S of X is closed set in (X,T) if its complement in X (X\S) is open in (X,T).

We call a set S is clopen if it is open and closed.

Example

Let us go back to the first example ,easily, we can find that;

{a} is clopen

{b,c} is un-clopen

{c,d} is open and not closed

{a,b,e,f} is closed and not open.

(1;3) Relationships Between Sets:[5,16]

Let f be a function from a set X into a set Y, then;

i. (1;3;1) The function f is said to be injective if f(x1)=f(x2) implies x1=x2, for all x1,x2 belong to X.
ii. (1;3;2) The function f is said to be surjective if for each y belong to Y there exists an x belongs to X such that f(x)=y.
iii. (1;3;3) The function f is said to be bijective if it is both surjective and injective.
iv. (1;3;4)The function f is said to have an inverse if there exists a function g of Y into X such that $g\big(f(x)\big) = x; x \in X \ and \ f\big(g(y)\big) = y; y \in Y$, and we call g an inverse function.

Corollaries (1;3;5)

i. The function f has an inverse if and only if f is bijective.
ii. There is just one inverse function between two sets.
iii. If g is an inverse function of f then f is an inverse function of g.

The inverse image of a set: (1;3;6)

Let $f: X \rightarrow Y \ and \ S \subset Y$, then the set $f^{-1}(S)$ is the inverse image of S, and it is defined by;

$$f^{-1}(S) = \{x: x \in X \wedge f(x) \in S\}$$

(1;4) Limit Points: [16]

Definition (1;4;1)

Let A be a subset of a topological space (X,T). A point x belongs to X is said to be a limit point of A if every open set containing x contains a point of A different from x.

- Example: Let (X,T) a topological space where X={a,b,c,d,e}, T={X,Ø,{a},{c,d},{a,c,d},{b,c,d,e}}, A={a,b,c}. Then b,d,e are limit points of A but a,c are not.
- Example: Let (X,T) be an indiscrete space and A a subset of X with at least 2 elements, then every point of X is a limit point of A.

(1;5) Connectedness: [16]

Definition (1;5;1)

Let (X,T) be a topological space. Then it is said to be connected if the only clopen sets of X are X and Ø.
So the space R is connected.

Lemma (1;5;2): let (X,T) be a topological space then it is not connected if and only if it has proper non-empty disjoint open subsets A,B such that $A \cup B = X$.

(1;6)Homeomorphisms:[9,16,20]

In each branch of mathematics it is essential to recognize when 2 structures are equivalent. For example 2 sets are equivalent if there exists a bijective function which maps one set into the other. And 2 topological spaces are equivalent, known as Homeomorphic, if there exists Homeomorphism of one onto the other.

- Example:
  X={a,b,c,d,e,f} , Y={g,h,i,j,k} ,
  T={X,Ø,{a},{c,d},{a,c,d},{b,c,d,e}} , T1={Y,Ø,{g},{i,j},{g,i,j},{h,i,j,k}}

It is clear that in an intuitive sense (X,T) is equivalent to (Y,T1), and the function $f: X \to Y$,defined by $f(a) = g, f(b) = h, f(c) = i, f(d) = j, f(e) = k$, provides the equivalence.

Corollary (1;6;1)

Let $f: X \to Y$ be a function and (X,T),(X,T1) are two toplogical spaces; then we call f homeomorphism if; $\forall U \in T1 \Longrightarrow f^{-1}(U) \in T$; if $f(x1) = f(x2) \Longrightarrow x1 = x2$ and $\forall y \in Y, \exists x \in X\ so\ that\ f(x) = y$.

And if there are two homeomorphic topological spaces (X,T),(Y,T1), then we write $(X,T) \cong (Y,T1)$.

Some properties(1;6;2):

i. $(X,T) \cong (X,T)$ so it is reflexive.
ii. $(X,T) \cong (Y,T1) implies (Y,T1) \cong (X,T)$ so it is symmetric, and its inverse is also a homeomorphism.
iii. $(X,T) \cong (Y,T1)\ and\ (Y,T1) \cong (Z,T2)\ implies\ (X,T) \cong (Z,T2)$ so it is transitive.
iv. Any topological space homeomorphic to a connected space is connected, so all the properties of a topological space which is homeomorphic to another one are in the other one.

Continuous Mappings(1;7): [7,8]

Definition (1;7;1)

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be continuous if:

$\forall a \in \mathbb{R}; \forall \varepsilon > 0; \exists \delta > 0; \text{if } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

In other words we can say if $f: \mathbb{R} \to \mathbb{R}$ then we call it continuous if and only if for $\forall a \in \mathbb{R}\ and\ for\ each\ interval\ (f(a) - \varepsilon, f(a) + \varepsilon)\ for\ \varepsilon > 0, there\ exists\ a\ \delta > 0$, $such\ that\ f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) for\ all\ x\ \in (a - \delta, a + \delta)$.

Let (X,T) and (Y,T1) be topological spaces and f a function from X into Y, Then $f: (X,T) \to (Y,T1)$ is said to be a continuous mapping if for each $U \in T1, f^{-1}(U) \in T$.

Lemmas:[16]

- (1;7;2) Let (X,T), (Y,T1) and (Z,T2) be topological spaces. If $f: (X,T) \to (Y,T1)\ and\ g: (Y,T1) \to (Z,T2)$ are continuous mappings, then $g \circ f: (X,T) \to (Z,T2)$ is continuous.

- (1;7;3) Let (X,T) and (Y,T1) be topological spaces, Then $f:(X,T) \to (Y,T1)$ is continuous if and only if for every closed subset S of Y, $f^{-1}(S)$ is a closed subset of X.
- (1;7;4) Let (X,T) and (Y,T1) be topological space and $f:(X,T) \to (Y,T1)$ is a continuous mapping , let $S \subset Y$, $S1 = Y\backslash S$ then $f^{-1}(S1) = X\backslash f^{-1}(S)$.
- (1;7;5) Let (X,T) and (Y,T1) be topological spaces and f a function from X into Y, Then $f:(X,T) \to (Y,T1)$ is a homeomorphism if and only if: f is continuous, the inverse exists and f^{-1} is continuous.
- (1;7;6) Let (X,T) and (Y,T1) be topological spaces and $f: X \to Y$ is a continuous mapping, A is a subset of X and T2 the induced topology on A, let $g:(A,T2) \to (Y,T1)$ be the restriction of f to A; that is g(x)=f(x) for all elements of A, then g is continuous.
- (1;7;7) Let (X,T) and (Y,T1) be topological spaces and $f: X \to Y$ is a continuous and surjective mapping, if (X,T) is connected then (Y,T1) is connected.

Metric Spaces(1;8):[16]

First we recall certain fundamental properties of real numbers: for all x,y,z ∈R,

1. |x−y|≥0;|x−y|=0 if, and only if, x =y;
2. |x−y|=|y−x|;
3. |x−y|≤|x−z|+|z−y|.

The quantity |x−y| is naturally thought of as the distance between the real numbers x and y. We seek to generalize all this, replacing R by an arbitrary non-empty set and |x−y| by a function of x and y which satisfies axioms based on 1,2 and 3. This is done, not simply as an exercise in the axiomatic approach, but because the structure obtained will enable us to solve many apparently different problems with the same technique.

Definition (1;8;1)

- Let X be a non-empty set and let d :X ×X →R be such that for all x,y,z ∈X,
  - (i) d(x,y) ≥0; d(x,y) =0 if, and only if, x =y;
  - (ii) d(x,y) =d(y,x) (the symmetry property);
  - (iii) d(x,y) ≤d(x,z)+d(z,y) (the triangle inequality). The function d is called a metric or distance function on X; the pair (X,d) is called a metric space; when no ambiguity is possible we shall,

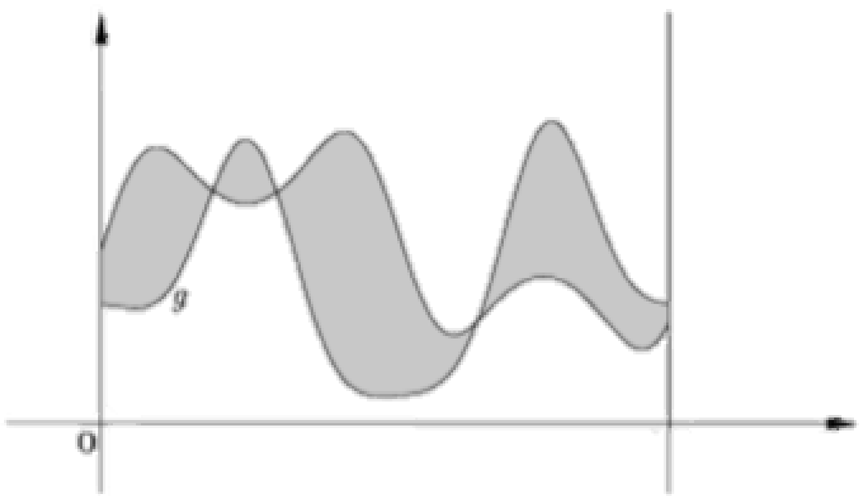
for simplicity, often refer to X, rather than (X,d), as a metric space.

- Let n be a positive integer:
  $X = R_n = \{x = (x_1,...,x_n) = (x_i) : x_i \in R$ for $i=\{1,...,n\}$.
  Various metrics can be defined on this set in a natural way: some of the most common are $d_p (1 \leq p < \infty)$ and d_∞, where
  $d_p(x,y) = (\sum_{i=1}^{1} |x_i - y_i|^p)^{\frac{1}{p}}$

Example: let L the set of continuous functions from [0,1] into R. A metric is defined on this set by:

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx$$

Where f and g are in L, then the region which lies between the graphs of the functions is illustrated below.



PS: $|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ because $|x-y| = d_2(x,y)$.
The generalization of the third condition(Minkowski's inequality):

$$\left(\sum_1^n |x_i - y_i|^p\right)^{1/p} \le \left(\sum_1^n |x_i|^p\right)^{1/p} + \left(\sum_1^n |y_i|^p\right)^{1/p}$$

$$\le \left(\sum_1^\infty |x_i|^p\right)^{1/p} + \left(\sum_1^\infty |y_i|^p\right)^{1/p}.$$

Open Balls(1;9): [7,8,9,16]

Definition (1;9;1)

Let (X,d) be a metric space. Given any $x \in X$ and any $r > 0$, let $B(x,r) = \{y \in X : d(x,y) < r\}$: $B(x,r)$ is called the open ball in X with centre x and radius r. A subset G of X is called open if given any $x \in G$, there exists $r > 0$ (depending upon x) such that $B(x,r) \subset G$.

Convergent sequences(1;10): [7,8]

1. A sequence (x_n) in a metric space (X,d) is said to converge to a point $x \in X$ if, and only if:
   $\forall \varepsilon \succ 0 : \exists n_\varepsilon \in \aleph; n \ge n_\varepsilon \Rightarrow d(x, x_n) \prec \varepsilon$
   We can write this as: $x_n \to x, \lim_{n\to\infty} x_n = x; d(x, x_n)_{n\to\infty} \to \infty$
2. A sequence (x_n) in X is said to be convergent if, and only if, there exists $x \in X$ such that $x_n \to x$; we also say in this case that (x_n) has limit x.
3. Let(x_n) be a sequence in a metric space (X,d).Then (x_n) converges to at most one point.

2-Henri Poincare[10,11]

Henri Poincaré's father was Léon Poincaré and his mother was Eugénie Launois. They were 26 and 24 years of age, respectively, at the time of Henri's birth. Henri was born in Nancy where his father was Professor of Medicine at the University.

Léon Poincaré's family produced other men of great distinction during Henri's lifetime.

Raymond Poincaré, who was prime minister of France several times and president of the French Republic during World War I, was the elder son of Léon Poincaré's brother Antoine Poincaré. The second of Antoine Poincaré's sons, Lucien Poincaré, achieved high rank in university administration.

Henri was ambidextrous and was nearsighted; during his childhood he had poor muscular coordination and was seriously ill for a time with diphtheria. He received special instruction from his gifted mother and excelled in written composition while still in elementary school.

In 1862 Henri entered the Lycée in Nancy (now renamed the Lycée Henri Poincaré in his honor). He spent eleven years at the Lycée and during this time he proved to be one of the top students in every topic he studied. Henri was described by his mathematics teacher as a "monster of mathematics" and he won first prizes in the concourse general, a competition between the top pupils from all the Lycées across France.

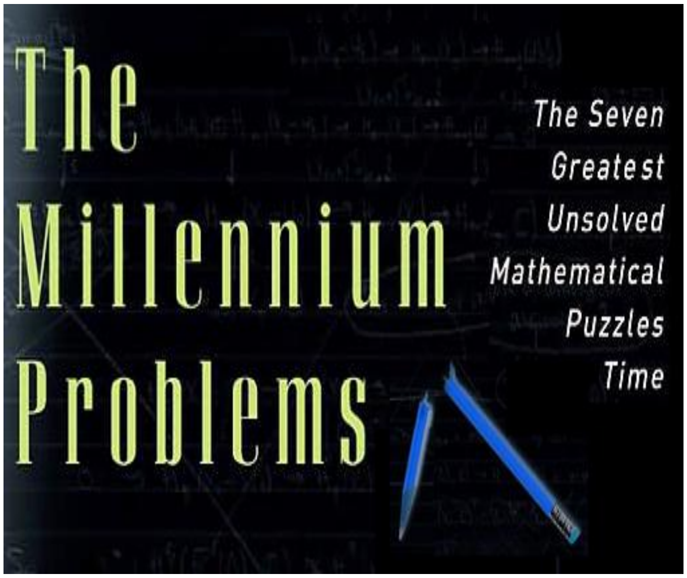
Poincaré entered the École Polytechnique in 1873, graduating in 1875. He was well ahead of all the other students in mathematics but, perhaps not surprisingly given his poor coordination, performed no better than average in physical exercise and in art. Music was another of his interests but, although he enjoyed listening to it, his attempts to learn the piano while he was at the École Polytechnique were not successful. Poincaré read widely, beginning with popular science writings and progressing to more advanced texts.

His memory was remarkable and he retained much from all the texts he read but not in the manner of learning by rote, rather by linking the ideas he was assimilating particularly in a visual way. His ability to visualize what he heard proved particularly useful when he attended lectures since his eyesight was so poor that he could not see the symbols properly that his lecturers were writing on the blackboard. After graduating from the École Polytechnique, Poincaré continued his studies at the École des Mines. Immediately after receiving his doctorate, Poincaré was appointed to teach mathematical analysis at the University of Caen. Reports of his teaching at Caen were not wholly complimentary, referring to his sometimes disorganized lecturing style. He was to remain there for only two years before being appointed to a chair in the Faculty of Science in Paris in 1881. In 1886 Poincaré was nominated for the chair of mathematical physics and probability at the Sorbonne. The intervention and the support of Hermite was to ensure that Poincaré was appointed to the chair and he also was appointed to a chair at the École Polytechnique.

Before the age of 30 he developed the concept of automorphic functions which are functions of one complex variable invariant under a group of transformations characterized algebraically by ratios of linear terms. The idea was to come in an indirect way from the work of his doctoral thesis on differential equations. His results applied only to restricted classes of functions and Poincaré wanted to generalize these results but, as a route towards this, he looked for a class functions where solutions did not exist. For 40 years after Poincaré published the first of his six papers on algebraic topology in 1894, essentially all of the ideas and techniques in the subject were based on his work. The Poincaré conjecture remained as one of the most baffling and challenging unsolved problems in algebraic topology until it was settled by Grisha Perelman in 2002.

Homotopy theory reduces topological questions to algebra by associating with topological spaces various groups which are algebraic invariants. Poincaré introduced the fundamental group (or first homotopy group) in his paper of 1894 to distinguish different categories of 2-dimensional surfaces. He was able to show that any 2-dimensional surface having the same fundamental group as the 2-dimensional surface of a sphere is topologically equivalent to a sphere. He conjectured that this result held for 3-dimensional manifolds and this was later extended to higher dimensions. Surprisingly proofs are known for the equivalent of Poincaré's conjecture for all dimensions strictly greater than three. No complete classification scheme for 3-manifolds is known so there is no list of possible manifolds that can be checked to verify that they all have different homotopy groups.

3-The Millennium Prize[10,11]



In David Hilbert's lecture in 1900, he presented 15 difficult problems that guided much of the research of mathematicians in the last century.

Today, only 12 of these 15 problems have been solved. In honor of this great mathematician and to celebrate mathematics of the new millennium, the Clay Mathematics Institute of Cambridge, Massachusetts is offering a one million dollar prize for the solution to any one of the "Millennium Prize Problems". These problems, like Hilbert's problems, are long-standing mathematical questions that still have not been solved after many years of serious attempts by different experts. For my honors thesis project, I chose to research and study the Poincare conjecture.

The Riemann Hypothesis is one of Hilbert's unsolved problems and is now widely regarded as the most important open problem in pure mathematics. The Riemann Hypothesis is a proposition about the zeros of the Riemann zeta function, developed by G.F.B. Riemann (1826-1866): $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ where s is a complex number.

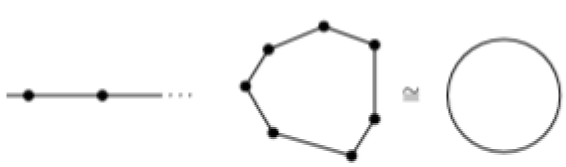
(3,1)The Poincare Conjecture

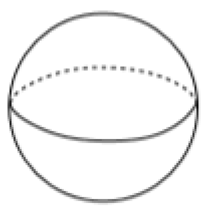
The classification of manifolds: [1,2,3,6,13]

Topology is the study of spaces under continuous deformation (the technical term is homeomorphism). The most important examples of topological spaces are manifolds. A manifold is a space which is locally like ordinary space of some particular dimension. Thus a 1–manifold is locally like a line, a 2–manifold is locally like a plane, a 3–manifold like ordinary 3–dimensional space and so on. In order to avoid pathological examples, it is usual to insist that a manifold has some additional structure, which tames it; the principal extra structures which are assumed are differential structure the ability to do analysis on the manifold and piecewise-linear (PL) structure, which is roughly equivalent to assuming that the manifold can be made of straight pieces. In low dimensions (less than seven) these two seemingly opposed structures are in fact equivalent, and we shall freely assume either of them as necessary.

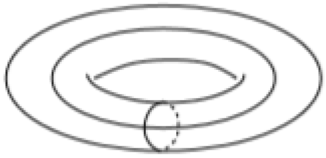
Let us explore manifolds by starting in low dimensions and working upwards. In dimension 1, there is very little to discuss. If we assume that our manifold has a PL structure, then it must consist of a number of intervals laid end to end. If we also assume that it is connected (all in one piece) then it either comprises an infinite collection of intervals forming a line or it comprises a finite collection forming a circuit. Thus a connected 1–manifold is homeomorphic either to a line or to a circle. Of these only the circle is compact (a technical notion equivalent to being made of a finite number of straight pieces). So there is just one compact, connected 1–manifold up to homeomorphism.

- The simplest 2–manifold is the 2–sphere (the surface of a ball).





- The next simplest and easily visualized is the torus (the surface of an American doughnut).



An infinite family of surfaces can now be described. Join two tori together by a tube to form a double torus. (This joining operation is called connected sum.)

(3;2)The conjecture: [4,9,10,12,13,14,15,17,18,19]

The Poincare Conjecture is that a (compact connected) 3manifold whose fundamental group is trivial is homeomorphic to S^3.

For 2manifolds(surfaces)the conjecture makes sense in exactly the same terms: every (compact connected) 2–manifold whose fundamental group is trivial is homeomorphic to S^2. Indeed looking at the list of 2–manifolds given earlier, it is easy to see that all have non-trivial fundamental groups apart from S2. So this is in fact a theorem not a conjecture. However if we move up to dimension 4 then it is easy to construct compact connected 4manifolds with trivial fundamental group which are not the 4sphere. One simple example is “complex projective space”, which is the complex analogue of the projective plane described earlier. Since this manifold plays no further role in this essay, I shall say no more about it.

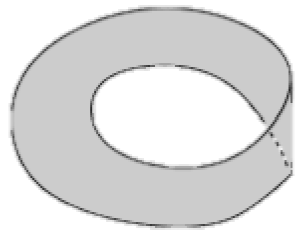
Moving up to higher dimensions, it is again easy to find compact connected manifolds with trivial fundamental group which are not the sphere of the appropriate dimension. Therefore to generalize the conjecture to higher dimensions sensibly we need to change it to an equivalent conjecture. The idea of “homotopy equivalence” is the key one.

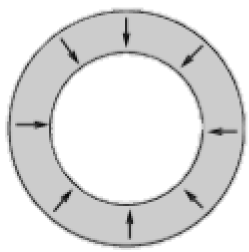
The concept of homeomorphism was touched on briefly before. A homeomorphism between two spaces is a continuous bijection between them with continuous inverse. The idea is that it may change sizes-it may deform the space-but it may not tear it apart. This is what continuity means.

The idea of homotopy equivalence is to allow general maps rather than bijections and ask that the compositions be merely homotopic to bijections rather than actually being bijections. The homotopy allows great flexibility and makes the relationship much more general.

For a simple example, consider an annulus. This can be squeezed down to a circle:

This squeezing is not a homeomorphism but it is a homotopy equivalence. In a similar way a Mobius band (below) can be deformed to a circle by squeezing onto the centre line. Thus all three (Mobius band, annulus and circle) are homotopy equivalent. Homotopy equivalence captures the global algebraic invariants of a space, whilst losing the local geometry.





Let us call an n–manifold a "homotopy sphere" if it is homotopy equivalent to S_n. Now it can be proved that a 3manifold is a homotopy sphere if and only if it is simply-connected(has trivial fundamental group). Thus the Poincare Conjecture can be reformulated: Every homotopy 3sphere is homeomorphic to S^3.

Conclusion

Just as in the case of our Earth, an atlas of the universe would be a collection of maps. But a map of a region of the universe would not be a rectangular piece of paper. Rather it would look like a solid glass box (think of an aquarium or transparent shoebox) filled with clear liquid crystals in which spots would be lit up to correspond to the positions of planets and stars. In the box-map that contained our solar system, if we looked at a distance corresponding to 431 light years straight up from the Earth, you would see a point corresponding to Polaris, the North Star. Looking away from the Sun in various directions in the plane of the Earth's orbit, we would see the other planets. South of the plane of the equator in one direction, at a distance corresponding to a little more than 4 light years, would be our closest neighbors, Proxima Centauri and the double star Alpha Centauri. Depending on the scale of the box-map, out in another direction, also south of the equator, there would be the galactic center, with its massive black hole, at a distance corresponding to 25,000lightyears. Further away in a somewhat different direction would be Andromeda, the spiral galaxy nearest to us, at a distance corresponding to 2.9 million light years away. An atlas of the universe would be a collection of such transparent shoeboxes with every region mapped in at least one box. If, as seems likely, the universe does not go on forever, the number of boxes necessary to make up a full atlas would be finite. However, we can't "see" the universe put together as a whole. If we had a complete atlas of the universe, so that every part of it was mapped, we could try to assemble our clear shoebox maps. Yet, just as there is not enough room on the plane to put together maps of our world into a globe, there is not enough room in ordinary space to fit all the shoebox maps together nicely. We have trouble picturing the shape of a universe as a whole. Moreover, it is not possible to get outside the universe. This is an important difference between the Earth and the universe. A rocket can leave the surface of the Earth, and we can look at the Earth from outside it. Since we see in three dimensions and since the surface of the Earth is two-dimensional, we can see our planet bending in a third direction and visualize its whole shape easily. Even if we could get outside the universe in an attempt to see what shape it had, since the universe is three-dimensional, we would need to be able to see in at least four dimensions to visualize the universe as a whole. As we shall see, this does not mean that the universe does not have a shape. It also does not mean that the universe cannot curve. It could have many different shapes; and it is virtually certain that the universe, just like the surface of the Earth, curves differently in different places. Although the universe is immensely larger than the Earth, the study of the universe has some distinct advantages unavailable to the heirs of Pythagoras as they puzzled about the surface of our planet. Unlike the Earth where our vision is cut off by a horizon so cramped that we have to travel to make a map that covers areas on ably large area, we can see for a long way into the universe using telescopes. We have also got quite good at measuring distances to the various objects we see in the skies. Thus, we can construct a map of a rather large region of the universe without actually traveling off the earth. Our mathematics is much further advanced than it was in Columbus's time, and there are many powerful mathematical tools that can be brought to bear on the question of the shape of the universe. So the Poincare conjecture was the most useful tool to understand the manifolds, the objects and the universe.

My final Result: if a manifold has a property that every simple closed curve can be deformed into a point so it is homeomorphic to the sphere because it is continuous, and if we take a sub-manifold we find it is homeomorphic to the sphere depending on Poincare Conjecture, since it is continuous, the whole manifold is homeomorphic to the sphere.

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